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Rapid Note

Quasi Bernoulli fluctuations in random and disordered systems

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Abstract. It is shown that quasi Bernoulli fluctuations, which appear at a morphological phase transition, can be considered as a statistical basis for multifractal processes with constant multifractal specific heat in a wide class of random and disordered systems. This class contains at least following processes: percolation, diffusion-limited aggregation and corrosion, Lorenz like attractors, and mesoscopic systems with Anderson transition.

PACS. 05.40. +j Fluctuation phenomena, random processes, and Brownian motion – 05.45. +b Theory and models of chaotic systems

It is shown in a recent paper [1] that a wide class of random and disordered systems can be identified as a thermodynamic class with constant multifractal specific heat. This class contains at least following systems: percolation, diffusion-limited aggregation and corrosion, Lorenz like attractors, and mesoscopic systems with Anderson transition. So that it is an interesting problem to understand statistical nature of this phenomenon. In this note we show that quasi Bernoulli fluctuations, which appear at a morphological phase transition, can be considered as a statistical basis for such thermodynamic phenomenon.

Suppose that the total volume of a sample consist of a d-dimensional cube of size L. We divide this volume into N boxes of linear size r ($N \sim (L/r)^d$). We label each box by the index i and construct for each box the measure of a field $\mu(\mathbf{x}, t)$ (function on space ($\mathbf{x}$) and on time (t) variables)

$$\mu_i(r) = \int_{v_i} \mu(\mathbf{x})\ d\mathbf{x} \tag{1}$$

where v_i is volume of the ith box. Then the generalized dimension, D_q, can be introduced by follows scaling relationship (see, for instance, [2] and references therein)

$$Z_p = \Sigma_{i=1}^{N} [\mu_i(r)]^p \sim r^{(p-1)D_p}. \tag{2}$$

Let us define

$$\overline{\mu_i} = \mu_i / \underset{i}{max}\{\mu_i\}. \tag{3}$$

Then

$$\langle \overline{\mu}^p \rangle = \frac{1}{N} \sum_i \overline{\mu_i}^p. \tag{4}$$

The simplest structure, that can be used for fractal description, is a system for which $\overline{\mu_i}$ can take only two values 0 and 1. It follows from equations (3, 4) that for such system (with $q > 0$)

$$\langle \overline{\mu}^p \rangle = \langle \overline{\mu} \rangle \tag{5}$$

and fluctuations in this system can be identified as Bernoulli fluctuations [3].

Generalization of equation (5) in form of a generalized scaling

$$\langle \overline{\mu}^p \rangle \sim \langle \overline{\mu} \rangle^{f(p)} \tag{6}$$

can be used to describe more complex systems. We use invariance of the generalized scaling (Eq. (6)) with dimension transform [4]

$$\overline{\mu_i} \to \overline{\mu_i}^{\lambda} \tag{7}$$

to find $f(p)$. This invariance means that

$$\langle (\overline{\mu}^{\lambda})^p \rangle \sim \langle (\overline{\mu}^{\lambda}) \rangle^{f(p)} \tag{8}$$

for all positive λ. Then, it follows from equations (6, 8) that

$$\langle (\overline{\mu})^{\lambda p} \rangle \sim \langle (\overline{\mu}) \rangle^{f(\lambda p)} \sim \langle (\overline{\mu}) \rangle^{f(\lambda) f(p)}. \tag{9}$$

Hence,

$$f(\lambda p) = f(\lambda) f(p). \tag{10}$$

The general solution of functional equation (10) is

$$f(p) = p^{\gamma} \tag{11}$$

where γ is a positive number (*cf.* [5]). It should be noted that case $\gamma = 1$ corresponds to Gauss fluctuations [6]. We, however, shall consider limit $\gamma \to 0$, *i.e.* transition to the Bernoulli fluctuations. This transition is non-trivial. Indeed, let us consider generalized scaling

$$F_{qm} \sim F_{km}^{\alpha(q,k,m)} \tag{12}$$

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where

$$F_{qm} = \langle \overline{\mu}^q \rangle / \langle \overline{\mu}^m \rangle. \tag{13}$$

Substituting equation (6) into equations (12, 13) and using equation (11) we obtain

$$\alpha(q,k,m) = \frac{q^\gamma - m^\gamma}{k^\gamma - m^\gamma}\cdot$$

Hence,

$$\lim_{\gamma \to 0} \alpha(q,k,m) = \frac{\ln(q/m)}{\ln(k/m)}\cdot \tag{14}$$

If there is ordinary scaling

$$\langle \overline{\mu}^p \rangle \sim (r/L)^{\zeta_p}, \tag{15}$$

then

$$\alpha(q,k,m) = \frac{\zeta_q - \zeta_m}{\zeta_k - \zeta_m}\cdot \tag{16}$$

From comparison equations (14, 16) it follows that at the limit $\gamma \to 0$

$$\frac{\zeta_q - \zeta_m}{\zeta_k - \zeta_m} = \frac{\ln(q/m)}{\ln(k/m)}\cdot \tag{17}$$

General solution of functional equation (17) is

$$\zeta_q = a + c \ln q, \tag{18}$$

where a and c are some constants.

If we use relationship

$$\max_i\{\mu_i\} \sim (r/L)^{D_\infty} \tag{19}$$

(see, for instance, [7]), then it follows from equations (2, 3) and equations (15, 18, 19) that

$$D_q = D_\infty + c\frac{\ln q}{(q-1)} \tag{20}$$

for the quasi Bernoulli fluctuations (*i.e.* at the limit $\gamma \to 0$).

In terms of a thermodynamic interpretation of multifractality described in [2] the constant c in equation (20) can be interpreted as a constant multifractal specific heat. It is shown in paper [1] that the constant specific heat approximation gives good fitting to data obtained for a wide class of random and disordered systems. Since in paper [1] data for a large number of these systems are compared with the constant specific heat approximation we will consider only one example in this note. Namely, in paper [8] the surface mass exponent β_n have been determined for two-dimensional percolation clusters by probing the surface of growing percolation clusters [9] using random walkers. Quantity $D_p = D\beta_{p-1}$ is suggested in [8] to estimate the generalized dimension D_p. Here D is the fractal dimension of the set on which the adequate (so-called harmonic) measure resides. Figure 1 shows the data obtained in [8] and axes in this Figure are chosen for comparison with equation (20) (straight line). One can see good agreement of representation (Eq. (20)) with the data for finite q. Moreover, $c \simeq 1/3$, calculated from this figure,

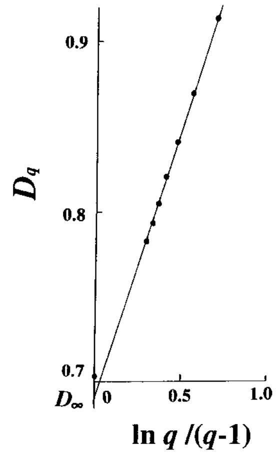

Fig. 1. Generalized dimensions D_q against $\ln(q)/(q-1)$ for 2D percolation clusters surface. Data (dots) are taken from [8]. The straight line is drawn for comparison with representation (Eq. (20)).

is the same as c calculated in [1] from the data of the two-dimensional simulations of diffusion-limited aggregation [10,11] and corrosion [12]. One can also see from Figure 1 that agreement between estimation of D_∞ obtained in this numerical simulation (a dot on the vertical axis) and value of D_∞ given by extrapolation of the representation (Eq. (20)) (intersection of the straight line corresponding to (Eq. (20)) with the vertical axis) is less good. If this (rather small) difference has not been caused by errors of the numerical simulation (or its interpretation) only, then it seems to be an interesting problem for future investigations. In any case, one can consider the quasi Bernoulli fluctuations as a possible statistical basis for the constant specific heat approximation in the multifractal thermodynamics.

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